

A FAMILY OF FINITE BALANCED p -GROUPS WITH UNBOUNDED NILPOTENCY CLASS

GROUPTHEORY50

ABSTRACT. For every prime p we construct a family of balanced finite two-generator p -groups whose nilpotency class is arbitrarily large. This gives an affirmative answer to Problem 17.113 in the Kourovka Notebook.

1. INTRODUCTION

The deficiency of a finite presentation $\mathcal{P} = \langle X \mid R \rangle$ is $\text{def}(\mathcal{P}) = |X| - |R|$, and the deficiency of a finitely presented group G is

$$\text{def}(G) = \sup\{\text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite presentation of } G\}.$$

We say that a presentation $\mathcal{P}$ is balanced if $\text{def}(\mathcal{P}) = 0$, and a group G is balanced if $\text{def}(G) = 0$.

Problem 17.113 of the Kourovka Notebook asks whether, for a fixed prime $p > 3$, there exist finite two-generator balanced p -groups with arbitrarily large nilpotency class [2]. Theorem 3.1 gives an affirmative answer to this question. We use the conventions $x^y = y^{-1}xy$ and $[x, y] = x^{-1}x^y$.

2. PRELIMINARIES

Presentations of metacyclic groups are studied in [1, §7.1]. There, the author proves the following result.

Theorem 2.1. *For $m, r, s, t \in \mathbb{N}$ such that $r^t \equiv 1 \pmod{m}$ and $s = m/\gcd(m, r-1)$, consider the group*

$$G = \langle a, b \mid a^m = 1, a^b = a^r, b^t = a^s \rangle.$$

Then G is a finite metacyclic group of order mt , and there exists an integer u such that G is presented by

$$\langle a, b \mid [b, a^{-u}] = a^{\gcd(m, r-1)}, b^t = a^s \rangle.$$

Any finite presentation of a finite group on $|X|$ generators requires at least $|S| \geq |X|$ relations [1, §6.5, Corollary 2]. Thus, every finite group has non-positive deficiency and a finite group is balanced if and only if it admits a balanced presentation. Therefore, the groups in Theorem 2.1 are balanced.

3. THE FAMILY

For a prime p and an integer $n \geq 1$, define

$$G_{p,n} = \langle a, b \mid a^{p^{n+1}} = 1, a^b = a^{1+p}, b^{p^n} = a^{p^n} \rangle.$$

Theorem 3.1. *For every $n \geq 1$ and every prime p , the group $G_{p,n}$ is a finite balanced p -group with nilpotency class $n+1$.*

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Proof. Since $\gcd(p^{n+1}, p) = p$ and $a^p = [b, a^{-1}] = [a, b]$, the integer u from Theorem 2.1 is 1, and $G_{p,n}$ is presented by

$$\langle a, b \mid [a, b] = a^p, a^{p^n} = b^{p^n} \rangle$$

From this, it follows that $G_{p,n}$ is a finite balanced p -group of order p^{2n+1} . The commutator relation gives $[a, b] = a^p$. Therefore, for $k \geq 0$,

$$[a^{p^k}, b] = a^{p^{k+1}}.$$

It follows inductively that

$$\gamma_r(G_{p,n}) = \langle a^{p^{r-1}} \rangle \quad (2 \leq r \leq n+2).$$

Thus $\gamma_{n+1}(G_{p,n}) \neq 1$ and $\gamma_{n+2}(G_{p,n}) = 1$, so the class is $n+1$. □

REFERENCES

- [1] D. L. Johnson., *Presentations of groups*, Cambridge University Press, 2012.
- [2] E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved Problems in Group Theory: The Kourovka Notebook*, No. 21, Novosibirsk, 2026; arXiv:1401.0300.