# nilcano

## Canonical conjugacy representatives in nilpotent groups

1.0

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# Chapter 1

# Preface

In this package we compute the canonical conjugacy representatives of elements and subgroups of nilpotent groups given by a nilpotent sequence. This package also include functions to compute the intersection of subgroups of nilpotent group and to compute the product of subgroups of nilpotent groups.

## 1.1 Nilpotent sequences

Let $G$ be a finitely generated nilpotent group. Then $G$ has a series $G = G_1 > G_2 > \ldots > G_n > G_{n+1} = 1$ so that each Gi is normal in G and each quotient $G_i/G_{i+1}$ is cyclic and central in $G/G_{i+1}$. We call such a series a nilpotent series of G. Then $(g_1, \ldots, g_n)$ is called a nilpotent sequence of $G$ and $(o_1, \ldots, o_n)$ are its relative orders. Note that the nilpotent sequence determines its nilpotent series as $G_i = \langle g_i, \ldots, g_n \rangle$ holds for $1 \leq i \leq n$.

Each element $g \in G$ can be written uniquely as $g = g_1^{e_1} \ldots g_n^{e_n}$ with $e_i \in \mathbb{Z}$ and $e_i \in \{0, \ldots, o_{i-1}\}$ if $i \in I$. The factorisation $g_1^{e_1} \ldots g_n^{e_n}$ for $g \in G$ is called the normal form of $g$. The associated integer vector $(e_1, \ldots, e_n)$ is the exponent vector of $g$. We write $e(g) = (e_1, \ldots, e_n)$. If $e_1 = \ldots = e_{i-1} = 0$ and $e_i \neq 0$, then we write $dep(g) = i$ and call this the depth of $g$. The leading exponent of an element $g$ is $e_d$ where $d = dep(g)$. The identity element satisfies $dep(1) = n + 1$ and does not have leading exponent.

# Chapter 2

# Conjugacy

## 2.1 Canonical conjugate representatives for elements

Suppose that the finitely generated group $G$ is given by a nilpotent sequence $(g_1,\ldots,g_n)$. For $g \in G$ and $U \leq G$, we denote the conjugacy class of $g$ under $U$ by $g^U = g^h | h \in U$ and the centralizer of $g$ in $U$ by $C_U(g) = \{h \in U | g^h = g\}$.

Let $\ll$ be the well-order $0 \ll 1 \ll 2 \ll \ldots \ll -1 \ll -2 \ldots$ of $\mathbb{Z}$. We order the exponent vectors extending the well-order lexicographically; that is, $(e_1,\ldots,e_n) \ll (f_1,\ldots,f_n)$ if $e_1 = f_1,\ldots,e_{i-1} = f_{i-1}$ and $e_i \ll f_i$ for some $i \in 1,\ldots,n$. We order the elements of G via their exponent vectors; that is, $g \ll h$ if $e(g) \leq e(h)$. One of the main program of this package is to compute the minimum element in $g^U$ with respect to $\ll$ this element is known as the canonical conjugacy representative of $g$, this element is denoted as $Cano_U(g)$. This program also computes the centralizer $C_U(g)$. This package is supplementary to the article of Eick and Fernández Ayala [EFA25].

### 2.1.1 CentralizerNilGroup

▷ `CentralizerNilGroup(`*G, elms*`)` (function)

Computes the centralizer of a given set of elements *elms* in *G*.

### 2.1.2 IsConjugateNilGroup

▷ `IsConjugateNilGroup(`*G, g, h*`)` (function)

Checks if two given elements $g$ and $h$ in $G$ are conjugate. If so it returns the conjugating element.

### 2.1.3 CanonicalConjugateElements

▷ `CanonicalConjugateElements(`*G, elms*`)` (function)

Computes the canonical conjugate representative of a given elements *elms* in *G*. Returns a record containing the canonical conjugates, the conjugating elements to obtain the canonical conjugate and the centralizers of the given elements.

### 2.1.4 IsCanonicalConjugateElements

▷ `IsCanonicalConjugateElements(G, elms)` (function)

Checks if a set of elements *elms* in $G$ are conjugate using the canonical representative aproach. If so it returns a record containing the canonical conjugate representative, the conjugating elements and the centralizer.

### 2.1.5 InfoConjugacyElements

▷ `InfoConjugacyElements` (info class)

Info class for the functions of canonical conjugates for elements.

## 2.2 Canonical conjugate representatives for subgroups

Suppose that the finitely generated group $G$ is given by a nilpotent sequence $(g_1,\ldots,g_n)$. Given $U,V \le G$ we write $U^V = \{U^g | g \in V\}$ for the conjugacy class of $U$ under $V$ and $N_V(U) = \{g \in V | U^g = U\}$ for the corresponding normalizer. Let $U \le G$ be given by its basis $(u_1,\ldots,u_n)$. Let $N \le N_G(U)$ and $g \in N$. We define $Cano_N^U(g)$ as the unique reduced preimage of $Cano_{N/U}(gU)$ under the natural homomorphism $N \to N/U$. Now we consider two subgroups $W$ and $V$ of $G$, and write $W_i = W \cap G_i$ for $1 \le i \le n$. We define $Cano_V(W)$ inductively: suppose that $U_{i+1} = Cano_V(W_i+1)$ is given by a basis $u_{i+1},\ldots,u_n$ together with a conjugating element $U_i+1 = W_{i+1}^v$ and its normalizer $N = N_V(U_{i+1})$. Suppose that $W_i \neq W_{i+1}$ and $W_i = \langle w_i, W_{i+1}\rangle$. Let $w_i'$ be the normalized power of the conjugate $w_i^v$. Set $u_i = Cano_N^{U_{i+1}}(w_i')$, then $(u_i,\ldots,u_n)$ is a basis for a subgroup $U_i$ of $G_i$. We write $Cano_V(W)$ for the subgroup $U_1$ eventually determined by an iterated process. One of the main program of this package is to compute $Cano_V(W)$. This program also computes the normalizer of $N_V(W)$.

### 2.2.1 NormalizerNilGroup

▷ `NormalizerNilGroup(G, U)` (function)

Computes the normalizer of a given subgroup $U$ of $G$.

### 2.2.2 IsConjugateSubgroups

▷ `IsConjugateSubgroups(G, U, V)` (function)

Checks if two given subgroups $U$ and $V$ of $G$ are conjugate. If so it returns the conjugating element.

### 2.2.3 CanonicalConjugateSubgroup

▷ `CanonicalConjugateSubgroup(G, U)` (function)

Computes the canonical conjugate representative of a given subgroup $U$ of $G$. Returns a record containing the canonical conjugate subgroup, the conjugating element and the normalizer of the given subgroup.

### 2.2.4 IsCanonicalConjugateSubgroups

▷ `IsCanonicalConjugateSubgroups(G, U, V)` (function)

Checks if two subgroup $U,V$ of $G$ are conjugate using the canonical representative aproach. If so it returns a record containing the canonical conjugate subgroup, the conjugating elements and the normalizer of the given subgroups.

### 2.2.5 InfoConjugacySubgroups

▷ `InfoConjugacySubgroups` (info class)

Info class for the functions of canonical conjugates for subgroups.

## 2.3 Canonical conjugate representatives for lists

### 2.3.1 CanonicalConjugateList

▷ `CanonicalConjugateList(G, list)` (function)

Given a list of elements of a nilpotent group $G$ returns a list of canonical representative conjugates of the list and the position of the elements that belong to the given canonical conjugacy class.

### 2.3.2 IsConjugateList

▷ `IsConjugateList(G, list)` (function)

Given a list of elements of a nilpotent group $G$ returns a list of representative conjugates of the list and the position of the elements that belong to the given conjugacy class.

## 2.4 Intersection of subgroups and product pairs of subgroups

This algorithms are based on Eddie's work [Lo98].

### 2.4.1 IntersectionSubgroupsNilGroups

▷ `IntersectionSubgroupsNilGroups(G, U, V)` (function)

Given two subgroups $U,V$ of a nilpotent group $G$ computes the intersection of both subgroups.

### 2.4.2 ProductDecomposition

▷ `ProductDecomposition(G, U, V, g)` (function)

Given two subgroups $U,V$ of a nilpotent group $G$ computes the subgroup product pair of both subgroups. Given two subgroups $U,V$ of a nilpotent group $G$ and an element $g$ in $G$ computes the decompostion of $g$ under the product pair of both subgroups.

# References


[EFA25] B. Eick and Ó. Fernández Ayala. The conjugacy problem and canonical representatives in finitely generated nilpotent groups. *Journal of Symbolic Computation*, 130:Paper No. 102422, 11, 2025. 4

[Lo98] E. Lo. Finding intersections and normalizers in finitely generated nilpotent groups. *Journal of Symbolic Computation*, 1:45 -- 59, 1998. 6

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