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Canonical conjugacy representatives in nilpotent groups

1.1

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Chapter 1

Preface

In this package we compute the canonical conjugacy representatives of elements and subgroups of nilpotent groups given by a nilpotent sequence. This package also includes functions to compute the intersection of subgroups of nilpotent groups and to compute the product of subgroups of nilpotent groups.

1.1 Nilpotent sequences

Let G be a finitely generated nilpotent group. Then G has a series $G = G_1 > G_2 > \dots > G_n > G_{n+1} = \{1\}$ so that each G_i is normal in G and each quotient G_i/G_{i+1} is cyclic and central in G/G_{i+1} . We call such a series a nilpotent series of G . Then $(g_1, \dots, g_n)$ is called a nilpotent sequence of G and $(o_1, \dots, o_n)$ are its relative orders. Let $I = \{i \in \mathbb{Z} \mid o_i \neq \infty\}$. Note that the nilpotent sequence determines its nilpotent series by $G_i = \langle g_i, \dots, g_n \rangle$ holds for $1 \leq i \leq n$.

Each element $g \in G$ can be written uniquely as $g = g_1^{e_1} \dots g_n^{e_n}$ with $e_i \in \mathbb{Z}$ and $e_i \in \{0, \dots, o_i - 1\}$ if $i \in I$. The factorisation $g_1^{e_1} \dots g_n^{e_n}$ for $g \in G$ is called the normal form of g . The associated integer vector $(e_1, \dots, e_n)$ is the exponent vector of g . We write $e(g) = (e_1, \dots, e_n)$. If $e_1 = \dots = e_{i-1} = 0$ and $e_i \neq 0$, then we write $\text{dep}(g) = i$ and call this the depth of g . The leading exponent of an element g is e_d where $d = \text{dep}(g)$. The identity element satisfies $\text{dep}(1) = n + 1$ and does not have leading exponent.

Chapter 2

Conjugacy

2.1 Canonical conjugate representatives for elements

Suppose that the finitely generated group G is given by a nilpotent sequence $(g_1, \dots, g_n)$. For $g \in G$ and $U \leq G$, we denote the conjugacy class of g under U by $g^U = \{g^h \mid h \in U\}$ and the centralizer of g in U by $C_U(g) = \{h \in U \mid g^h = g\}$.

Let $\ll$ be the well-order $0 \ll 1 \ll 2 \ll \dots \ll -1 \ll -2 \dots$ of $\mathbb{Z}$. We extend the well-order to exponent vectors lexicographically; that is, $(e_1, \dots, e_n) \ll (f_1, \dots, f_n)$ if $e_1 = f_1, \dots, e_{i-1} = f_{i-1}$ and $e_i \ll f_i$ for some $i \in 1, \dots, n$. We order the elements of G via their exponent vectors; that is, $g \ll h$ if $e(g) \leq e(h)$. One of the main programs of this package is to compute the minimum element in g^U with respect to $\ll$; this element is known as the canonical conjugacy representative of g , which is denoted as $\text{Cano}_U(g)$. This program also computes the centralizer $C_U(g)$. This package is supplementary to the article of Eick and Fernández Ayala [EF25].

2.1.1 CentralizerNilGroup

▷ `CentralizerNilGroup( $G$ ,  $elms$ )` (function)

Computes the centralizer of a given set of elements $elms$ in G .

2.1.2 IsConjugateNilGroup

▷ `IsConjugateNilGroup( $G$ ,  $g$ ,  $h$ )` (function)

Checks if two given elements g and h in G are conjugate. If so it returns the conjugating element.

2.1.3 CanonicalConjugateElements

▷ `CanonicalConjugateElements( $G$ ,  $elms$ )` (function)

Computes the canonical conjugate representative of a given elements $elms$ in G . Returns a record containing the canonical conjugates, the conjugating elements to obtain the canonical conjugate and the centralizers of the given elements.

2.1.4 IsCanonicalConjugateElements

▷ IsCanonicalConjugateElements(G , $elms$) (function)

Checks if a set of elements $elms$ in G are conjugate using the canonical representative approach. If so it returns a record containing the canonical conjugate representative, the conjugating elements and the centralizer.

2.1.5 InfoConjugacyElements

▷ InfoConjugacyElements (info class)

Info class for the functions of canonical conjugates for elements.

2.2 Canonical conjugate representatives for subgroups

Suppose that the finitely generated group G is given by a nilpotent sequence $(g_1, \dots, g_n)$. Given $U, V \leq G$ we write $U^V = \{U^g \mid g \in V\}$ for the conjugacy class of U under V and $N_V(U) = \{g \in V \mid U^g = U\}$ for the corresponding normalizer. Let $U \leq G$ be given by its basis $(u_1, \dots, u_n)$. Let $N \leq N_G(U)$ and $g \in N$. We define $\text{Cano}_N^U(g)$ as the unique reduced preimage of $\text{Cano}_{N/U}(gU)$ under the natural homomorphism $N \rightarrow N/U$. Now we consider two subgroups W and V of G , and write $W_i = W \cap G_i$ for $1 \leq i \leq n$. We define $\text{Cano}_V(W)$ inductively: suppose that $U_{i+1} = \text{Cano}_V(W_i + 1)$ is given by a basis $u_{i+1}, \dots, u_n$ together with a conjugating element $U_i + 1 = W_{i+1}^v$ and its normalizer $N = N_V(U_{i+1})$. Suppose that $W_i < W_{i+1}$ and $W_i = \langle w_i, W_{i+1} \rangle$. Let w_i' be the normalized power of the conjugate w_i^v . Set $u_i = \text{Cano}_N^{U_{i+1}}(w_i')$, then $(u_i, \dots, u_n)$ is a basis for a subgroup U_i of G_i . We write $\text{Cano}_V(W)$ for the subgroup U_1 eventually determined by an iterated process. One of the main programs of this package is to compute $\text{Cano}_V(W)$. This program also computes the normalizer of $N_V(W)$.

2.2.1 NormalizerNilGroup

▷ NormalizerNilGroup(G , U) (function)

Computes the normalizer of a given subgroup U of G .

2.2.2 IsConjugateSubgroups

▷ IsConjugateSubgroups(G , U , V) (function)

Checks if two given subgroups U and V of G are conjugate. If so it returns the conjugating element.

2.2.3 CanonicalConjugateSubgroup

▷ CanonicalConjugateSubgroup(G , U) (function)

Computes the canonical conjugate representative of a given subgroup U of G . Returns a record containing the canonical conjugate subgroup, the conjugating element and the normalizer of the given subgroup.

2.2.4 IsCanonicalConjugateSubgroups

▷ IsCanonicalConjugateSubgroups(G , U , V) (function)

Checks if two subgroup U, V of G are conjugate using the canonical representative approach. If so it returns a record containing the canonical conjugate subgroup, the conjugating elements and the normalizer of the given subgroups.

2.2.5 InfoConjugacySubgroups

▷ InfoConjugacySubgroups (info class)

Info class for the functions of canonical conjugates for subgroups.

2.3 Canonical conjugate representatives for lists

2.3.1 CanonicalConjugateList

▷ CanonicalConjugateList(G , $list$) (function)

Given a list of elements of a nilpotent group G , returns a list of canonical conjugacy representatives of the list and the positions of the elements that belong to the given canonical conjugacy class.

2.3.2 IsConjugateList

▷ IsConjugateList(G , $list$) (function)

Given a list of elements of a nilpotent group G , returns a list of conjugacy representatives of the list and the positions of the elements that belong to the given conjugacy class.

2.4 Intersection of subgroups and product pairs of subgroups

These algorithms are based on Eddie's work [Lo98].

2.4.1 IntersectionSubgroupsNilGroups

▷ IntersectionSubgroupsNilGroups(G , U , V) (function)

Given two subgroups U, V of a nilpotent group G computes the intersection of both subgroups.

2.4.2 SubgroupProductPair

▷ SubgroupProductPair(G , U , V) (function)

Given two subgroups U, V of a nilpotent group G , computes the subgroup product pair of both subgroups.

2.4.3 ProductDecomposition

▷ `ProductDecomposition( $G$ ,  $U$ ,  $V$ ,  $g$ )` (function)

Given two subgroups U, V of a nilpotent group G and an element g in G computes the decomposition of g with respect to the product pair of the two subgroups.

References

- [EF25] B. Eick and Ó. Fernández Ayala. The conjugacy problem and canonical representatives in finitely generated nilpotent groups. *Journal of Symbolic Computation*, 130:Paper No. 102422, 11, 2025. [4](#)
- [Lo98] E. Lo. Finding intersections and normalizers in finitely generated nilpotent groups. *Journal of Symbolic Computation*, 1:45 -- 59, 1998. [6](#)

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